

Estimation of the principle curvatures of approximated surfaces

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Received October 1999; revised February 2000

Abstract

This paper presents a method for estimating curvature values of a surface, which is given only approximatively, e.g., by measured data. The presented method requires estimates of the curvature of curves. These lead, along with the theorems from Euler and Meusnier, to the values of surface curvature. Methods are presented, which converge with the different error order of $O(h^2)$ and $O(h^{2n})$. For the first of them explicit formulae are given. It is proved that the error order remains the same through all further calculations until the final estimation of surface curvature is found. © 2000 Published by Elsevier Science B.V. All rights reserved.

Keywords: Curvature; Surface; Estimation

1. Basics

First, two results from differential geometry should be mentioned which we will use in the following sections: Meusnier's and Euler's theorem.

Let κ_v be the normal curvature of a curve c on a surface at a given point, κ_c the curvature of the curve and ϑ the angle between the plane in which the curve lies locally and the surface normal. A relation between these three values is given by Meusnier's theorem (do Carmo, 1993):

$$\kappa_c = \kappa_v \cos \vartheta.$$

Let κ_1 and κ_2 be the principle curvatures of a surface, φ the angle between the curve and the first direction of the principle curvature that belongs to κ_1 and κ_v the normal curvature of the curve. Euler's theorem provides a relation between them (do Carmo, 1993):

$$\kappa_v = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi.$$

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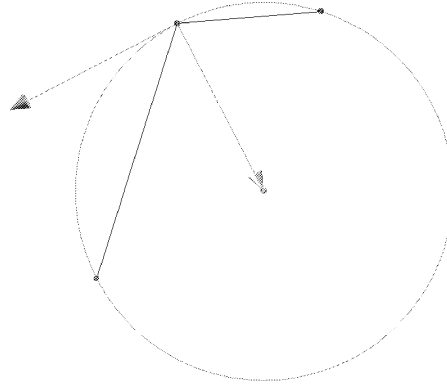


Fig. 1. Estimation of curvature and tangent with a circle.

2. Curvature estimation with a circle

Todd and McLeod (1986) use a circle to estimate the curvature of a curve approximated with three points. From this circle one can directly calculate the curvature and the tangent. One can easily prove that these are approximations of the curvature and tangent of the real curve (Fig. 1).

3. Curvature estimation with a Bézier segment

It is simple to imagine that estimating a curve approximated by three points p_0 , p_1 , p_2 with a circle, we can make big errors, especially if the angle in p_1 is very sharp. Therefore, we can use a quadratic Bézier curve B . Let

$$B(0) = p_0, \quad B(1/2) = p_1 \quad \text{and} \quad B(1) = p_2.$$

This means that the control points are

$$b_0 = p_0, \quad b_1 = \frac{2p_1 - (p_0 + p_2)}{2} \quad \text{and} \quad b_2 = p_2.$$

The derivatives at p_1 are $\dot{B}(1/2) = p_2 - p_0$ and $\ddot{B}(1/2) = p_2 + p_0 - 2p_1$. The curvature of a general regular curve is

$$\kappa_c = \frac{\sqrt{|\dot{c}|^2 |\ddot{c}|^2 - \langle \dot{c}, \ddot{c} \rangle^2}}{|\dot{c}|^3}.$$

Thus, the curvature of the Bézier segment in p_1 is

$$\kappa_B = \frac{4\sqrt{|p_2 + p_0 - 2p_1|^2 |p_2 - p_0|^2 - \langle p_2 - p_0, p_2 + p_0 - 2p_1 \rangle^2}}{|p_2 - p_0|^3}. \quad (1)$$

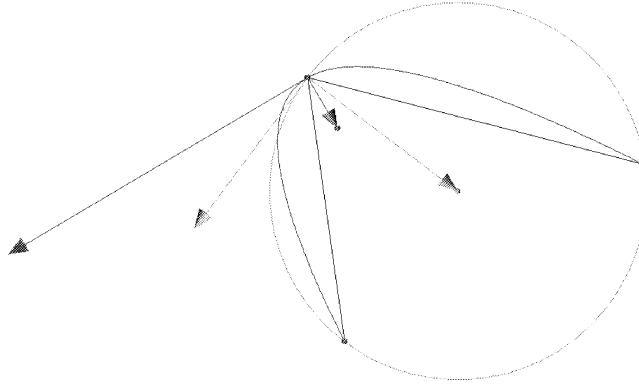


Fig. 2. Estimation of curvature and tangent with a Bézier segment.

One must prove that this is really an approximation of the curvature of our given curve. To prove this, we assume that the three points all lie on a curve c and this yields $c(0) = p_1$, $c(-h) = p_0$, $c(h) = p_2$. Performing transformations one obtains

$$\frac{\sqrt{\left| \frac{c(h)+c(-h)-2c(0)}{h^2} \right|^2 \left| \frac{c(h)-c(-h)}{2h} \right|^2 - \left(\frac{c(h)-c(-h)}{2h}, \frac{c(h)+c(-h)-2c(0)}{h^2} \right)^2}}{\left| \frac{c(h)-c(-h)}{2h} \right|^3} \quad (2)$$

One has to agree that

$$\lim_{h \downarrow 0} \frac{c(h) - c(-h)}{2h} = \dot{c}(0) \quad \text{and} \quad \lim_{h \downarrow 0} \frac{c(h) + c(-h) - 2c(0)}{h^2} = \ddot{c}(0),$$

and substitute this in the upper formula (2). Thus, we get the curvature $\kappa_c(0)$ of the curve c for $h \rightarrow 0$. Therefore, we can call the obtained value an estimation of the curvature of c in p_1 (Fig. 2).

4. Estimation of surface curvature with $O(h^2)$

The next goal is an estimate for surface curvature with error order $O(h^2)$. We use the theorems of Euler and Meusnier, so we need at least three directions with their corresponding normal curvature at the point we are interested in. First, we have to estimate all the values like derivations, surface normal, etc. with the same error order of $O(h^2)$. So we assume to have a given surface with a mesh of parameter lines and also with the diagonals.

Note that this is a not a strong restriction. We want to estimate the curvature in a point of a given mesh of triangles, and points with less than three directions do not appear often. Therefore, these points are not analyzed in this paper and we can locally assume that two of these three directions are parameter lines and the third is the diagonal.

First, we study the tangent and the curvature estimated with the Bézier segment. To see that these converge with order $O(h^2)$, we consider the Taylor series expansion of a function f in $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + O(x^4),$$

and after a transformation we recognize that estimating the tangent at $f(0)$ with the secant between $f(0)$ and $f(x)$ we get an error of $O(x)$, since

$$f'(0) = \frac{f(x) - f(0)}{x} + \frac{1}{2}f''(0)x + \dots$$

However, if we choose the secant between $f(-x)$ and $f(x)$ the error changes to $O(x^2)$. Therefore, we consider the same Taylor series expansion in $-x$:

$$f(-x) = f(0) - f'(0)x + \frac{1}{2}f''(0)x^2 - \frac{1}{6}f'''(0)x^3 + O(x^4).$$

After a transformation we get

$$f'(0) = \frac{f(x) - f(-x)}{2x} + \frac{1}{12}f'''(0)x^2 + \dots,$$

which was to be proved. For the estimation of the curvature we consider the second derivative and use the same principle which yields

$$f''(0) = \frac{f(x) + f(-x) - 2f(0)}{x^2} + O(x^2).$$

Thus, it is proved that the formula (1) we used to estimate the second derivative converges with $O(x^2)$.

Using curves $c(t)$ instead of functions $f(x)$ like above, one has to perform the calculations above for three coordinates.

For further study we need to know how the error order of $O(h^2)$ behaves in some basic operations. For $\alpha \in \mathbb{N}$, one obtains

$$(x + O(h^2))^\alpha = x^\alpha + O(h^2),$$

and for sufficiently small h :

$$\begin{aligned} (x + O(h^2))^{-\alpha} &= \frac{1}{x^\alpha} \left(\frac{1}{1 + \frac{O(h^2)}{x}} \right)^\alpha = \frac{1}{x^\alpha} \left(1 - \frac{O(h^2)}{x} \pm \dots \right)^\alpha \\ &= x^{-\alpha} + O(h^2). \end{aligned}$$

For $\alpha \in \mathbb{R} - \mathbb{Z}$ we consider the Taylor series expansion to see that it also yields

$$(x + O(h^2))^\alpha = x^\alpha + O(h^2).$$

Since scalar products are bilinear, it is

$$\langle x + O(h^2), y + O(h^2) \rangle = \langle x, y \rangle + O(h^2).$$

In the scalar product, $O(h^2)$ is a vector that has components of this order. Analogously it yields for the length of a vector

$$|x + O(h^2)| = \sqrt{\langle x + O(h^2), x + O(h^2) \rangle} = \sqrt{|x|^2 + O(h^2)} = |x| + O(h^2).$$

With these considerations it follows that

$$\begin{aligned} & \frac{\sqrt{\left| \frac{c(h)+c(-h)-2c(0)}{h^2} \right|^2 \left| \frac{c(h)-c(-h)}{2h} \right|^2 - \left\langle \frac{c(h)-c(-h)}{2h}, \frac{c(h)+c(-h)-2c(0)}{h^2} \right\rangle^2}}{\left| \frac{c(h)-c(-h)}{2h} \right|^3} \\ &= \frac{\sqrt{|\dot{c} + O(h^2)|^2 |\ddot{c} + O(h^2)|^2 - \langle \dot{c} + O(h^2), \ddot{c} + O(h^2) \rangle^2}}{|\dot{c} + O(h^2)|^3} \\ &= \dots = \frac{\sqrt{|\dot{c}|^2 |\ddot{c}|^2 - \langle \dot{c}, \ddot{c} \rangle^2}}{|\dot{c}|^3} + O(h^2) \\ &= \kappa_c(0) + O(h^2). \end{aligned}$$

For the principle normal we consider

$$\frac{\ddot{c} + O(h^2) - \langle \ddot{c} + O(h^2), \mathbf{t} + O(h^2) \rangle (\mathbf{t} + O(h^2))}{|\dots|} = \mathbf{h} + O(h^2).$$

We are now able to estimate curvature, tangent, and principle normal of a curve with an error order of $O(h^2)$.

Regarding surface curvature it is important to know the surface normal. We assumed to have a mesh of parameter lines, so we get the surface normal from the estimates of the tangents of the parameter lines:

$$\frac{(X_u + O(h^2)) \times (X_v + O(h^2))}{|(X_u + O(h^2)) \times (X_v + O(h^2))|} = \frac{X_u \times X_v}{|X_u \times X_v|} + O(h^2) = N + O(h^2).$$

Now we can apply the formula of Meusnier. We consider the angle ϑ between the principle normal \mathbf{h} and the surface normal N of the surface. It follows that

$$\langle \mathbf{h} + O(h^2), N + O(h^2) \rangle = \cos \vartheta + O(h^2),$$

and by applying the formula of Meusnier we obtain

$$(\kappa_\vartheta + O(h^2)) (\cos \vartheta + O(h^2)) = \kappa_\vartheta \cos \vartheta + O(h^2) = \kappa_N + O(h^2).$$

It is obvious that the three tangential directions to the surface that we need deviate from the tangent plane with $O(h^2)$ if we estimate them with help of a Bézier curve.

We have determined three normal curvatures with corresponding directions and angles between them. With an error order of $O(h^2)$ for the directions, the cosine of the angles as the scalar product between them has the same error. Assuming that the angle between the first direction and a direction of principle curvature is φ , we get three equations by using the formula of Euler:

$$\begin{aligned} k_1 + O(h^2) &= \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi = \kappa_1 + (\kappa_2 - \kappa_1) \sin^2 \varphi, \\ k_2 + O(h^2) &= \kappa_1 \cos^2(\varphi + t) + \kappa_2 \sin^2(\varphi + t) = \kappa_1 + (\kappa_2 - \kappa_1) \sin^2(\varphi + t), \\ k_3 + O(h^2) &= \kappa_1 \cos^2(\varphi + s) + \kappa_2 \sin^2(\varphi + s) = \kappa_1 + (\kappa_2 - \kappa_1) \sin^2(\varphi + s), \end{aligned} \tag{E}$$

where φ , κ_1 and κ_2 are the values of interest. This looks like a quadratic system of equations. However, it is explicitly solvable, and the solution is not complicated. We also

preserve the error order of $O(h^2)$ because we only use the basic calculations for the solution which have already been proved not to change the error order. By using a trigonometric addition theorem and making the substitutions $x := k_1 - \kappa_1$ and $y := \kappa_2 - k_1$ we get

$$\begin{aligned} k_3 - k_1 + (x - y) \sin^2 s &= \pm 2\sqrt{xy} \sin s \cos s + O(h^2), \\ k_2 - k_1 + (x - y) \sin^2 t &= \pm 2\sqrt{xy} \sin t \cos t + O(h^2). \end{aligned}$$

We can assume that $\sin t$ and $\sin s$ are not both 0, as well as $\cos s$ and $\cos t$ are not both 0. Otherwise a regular mesh of parameter lines would not exist because two of the three directions would be the same. Let

$$\begin{aligned} a &:= \frac{(k_2 - k_1) \sin s \cos s - (k_3 - k_1) \sin t \cos t}{\sin^2 s \sin t \cos t - \sin^2 t \sin s \cos s} = x - y, \\ b &:= \pm \frac{(k_2 - k_1) \sin^2 s - (k_3 - k_1) \sin^2 t}{2(\sin t \cos t \sin^2 s - \sin s \cos s \sin^2 t)} = \pm \sqrt{xy}, \end{aligned}$$

then we calculate by assuming $x > 0$ and $y > 0$:

$$x = \frac{1}{2}[a + \sqrt{a^2 + 4b^2}] + O(h^2) \quad \text{and} \quad y = \frac{1}{2}[-a + \sqrt{a^2 + 4b^2}] + O(h^2).$$

5. Estimation of surface curvature with $O(h^{2n})$

The next goal is to derive an estimation of surface curvature with an error order of $O(h^{2n})$. Again, we consider the Taylor series expansion of a function f at $x_0 = 0$:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

Let

$$g(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \dots$$

and

$$u(x) = a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \dots$$

Then, we calculate g from f as

$$g(x) = \frac{f(x) + f(-x)}{2}$$

and u as

$$u(x) = \frac{f(x) - f(-x)}{2}.$$

Estimating the first and second derivatives with the error order of $O(h^{2n})$ is equivalent to estimating a_1 and a_2 with the same error order. We add more points on a given curve and assume that they correspond to the parameter values $t, -t, 2t, -2t, \dots$. We can thus calculate

$$g(2x) = a_0 + 2^2a_2x^2 + 2^4a_4x^4 + 2^6a_6x^6 + 2^8a_8x^8 + \dots$$

and

$$u(2x) = 2^1 a_1 x + 2^3 a_3 x^3 + 2^5 a_5 x^5 + 2^7 a_7 x^7 + 2^9 a_9 x^9 + \dots,$$

...

$$g(nx) = a_0 + n^2 a_2 x^2 + n^4 a_4 x^4 + n^6 a_6 x^6 + n^8 a_8 x^8 + \dots$$

and

$$u(nx) = n^1 a_1 x + n^3 a_3 x^3 + n^5 a_5 x^5 + n^7 a_7 x^7 + n^9 a_9 x^9 + \dots$$

With $g(x)$ and $u(x)$ there are n equations from which we can form a linear combination so that the terms with x^4, x^6, \dots, x^{2n} and $x^3, x^5, \dots, x^{2n+1}$ vanish. We multiply the first equation with b_1 , the second with b_2 continuing to the last with b_n , so we get a system of equations with $n - 1$ equations:

$$b_1 + 2^4 b_2 + 3^4 b_3 + \dots + n^4 b_n = 0 \quad (1),$$

$$b_1 + 2^6 b_2 + 3^6 b_3 + \dots + n^6 b_n = 0 \quad (2),$$

...

$$b_1 + 2^{2n} b_2 + 3^{2n} b_3 + \dots + n^{2n} b_n = 0 \quad (n - 1).$$

We can determine a nontrivial solution for this linear system of equations in b_1, \dots, b_n , but the solution is not unique. If there is an additional equation, only the trivial solution is valid. This can be proved with methods relating to the Vandermonde determinant.

Let b_1, \dots, b_n solve the linear system of equations, then we can calculate $f''(x)$ as follows:

$$f''(x) = 2a_2 = 2 \frac{b_1 g(x) + b_2 g(2x) + \dots + b_n g(nx) - a_0(b_1 + \dots + b_n)}{x^2(b_1 + 2^2 b_2 + \dots + n^2 b_n)} + O(x^{2n}),$$

and we set

$$\frac{B}{x^2} + O(h^{2n}) := f''(x).$$

Analogously we compute a value for $f'(x)$:

$$f'(x) = a_1 = \frac{c_1 u(x) + c_2 u(2x) + \dots + c_n u(nx)}{x(c_1 + 2c_2 + 3c_3 + \dots + nc_n)} + O(x^{2n}) =: \frac{A}{x} + O(h^{2n}).$$

Similar to the method with $O(h^2)$ the error order of $O(h^{2n})$ stays the same through all basic calculations, so we have to analyze the expression:

$$\kappa = \frac{\sqrt{|\dot{c}|^2 |\ddot{c}|^2 - \langle \dot{c}, \ddot{c} \rangle^2}}{|\dot{c}|^3} = \frac{\sqrt{\frac{|A|^2 |B|^2}{x^2 x^4} - \langle \frac{A}{x}, \frac{B}{x^2} \rangle^2}}{\frac{|A|^3}{|x|^3}} = \frac{\sqrt{|A|^2 |B|^2 - \langle A, B \rangle^2}}{|A|^3}.$$

The estimation is given only in A and B , and these values only depend on the curve points and no longer on the parameters $x, -x, 2x, -2x, \dots$, which we needed to prove that the error order is $O(h^{2n})$. Therefore, the initial problem is reduced to solve a linear system of equations.

The obtained estimations of tangent and curvature of a curve can be used to estimate the values of surface curvature with the same error order, because it is proved that by solving the system of equations, obtained by the formula of Euler, error order will not change.

A possible algorithm for the $O(h^{2n})$ -method is:

- (1) Solve the linear system of equations in b_1, \dots, b_n and c_1, \dots, c_n .
- (2) For each of the three curve directions:
 - (2.1) Compute $g(x), \dots, g(nx)$ and $u(x), \dots, u(nx)$ from the data points.
 - (2.2) Compute

$$A := \frac{c_1 u(x) + c_2 u(2x) + \dots + c_n u(nx)}{c_1 + 2c_2 + 3c_3 + \dots + nc_n}.$$

- (2.3) Compute

$$B := 2 \frac{b_1 g(x) + b_2 g(2x) + \dots + b_n g(nx) - a_0(b_1 + \dots + b_n)}{b_1 + 2^2 b_2 + \dots + n^2 b_n}.$$

- (2.4) Compute the curvature

$$\kappa = \frac{\sqrt{|A|^2 |B|^2 - \langle A, B \rangle^2}}{|A|^3}.$$

- (2.5) Compute the tangent $T = A/|A|$.

- (3) Compute the angles t and s between the tangents of neighbouring curves.
- (4) Solve the system of equations (E) in Section 4.

We can also examine experimentally that the methods converge with the desired error order. We take an exact surface, calculate the values of surface curvature exactly and then apply our method of estimation on a given mesh on this surface. Let N be the estimator and have the error order $O(h^n)$, K be the exact value. There exists a constant c that yields

$$N = K + ch^n.$$

We choose an h_0 as initial segmentation and halve this value at every step, so we obtain the i th estimate

$$N_i = K + c \frac{h_0^n}{2^i}.$$

Further more, we consider

$$N_i - N_{i+1} = \frac{ch_0^n}{2^{n(i+1)}} (2^n - 1).$$

We neglect the -1 and get with a constant α :

$$\frac{\ln(N_i - N_{i+1})}{\ln 2} = \alpha - ni.$$

This means that if we link these values to each other in a cartesian coordinate system, we should get an approximated line with gradient $-n$.

In the following example α was subtracted so that the differences begin at zero. The surface is an ellipsoid, and the length of the main axes are 2, 6 and 10. The data point is $(\cos u_0 \cos v_0, 3 \sin u_0 \cos v_0, 5 \sin v_0)|_{u_0=0.3, v_0=0.5}$, and $h_0 = 0.2$. K is the Gaussian curvature, and the error order is $n = 2$ (Fig. 3).

I got the values:

$$\begin{aligned} &0.000000, \quad -1.961723, \quad -3.952360, \quad -5.950031, \quad -7.949451, \\ &-9.949306, \quad -11.949286, \quad -13.948441, \quad -15.963468, \quad -18.412731, \dots \end{aligned}$$

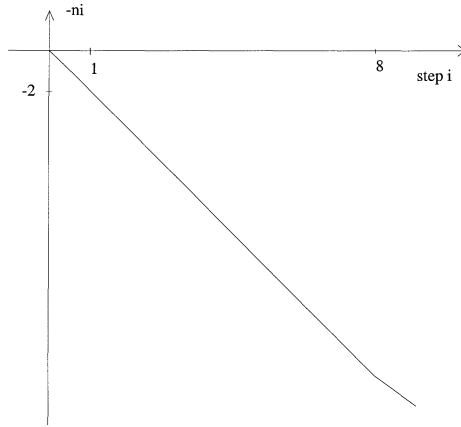


Fig. 3.

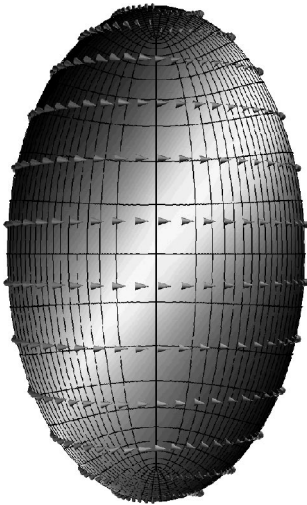


Fig. 4.

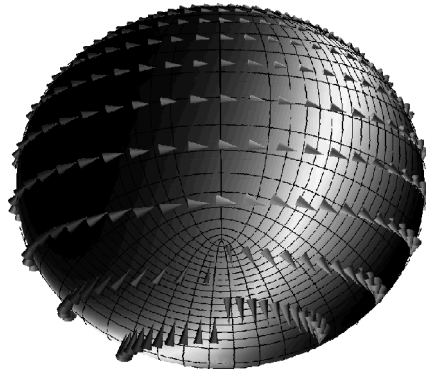


Fig. 5.

The following values have strongly deviated due to the errors made in the many calculations with the floating point variables (double).

The next image (Fig. 4) shows an ellipsoid with its exact lines of curvature and some estimated directions of principle curvature. The length of the three main axes are 2, 6 and 10.

Acknowledgements

The author thanks Professor Dr. W.L.F. Degen for the valuable suggestions.

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